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D-branes in field theory

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Abstract: Certain gauge theories in four dimensions are known to admit semi-classical D-brane solitons. These are domain walls on which vortex flux tubes may end. The purpose of this paper is to develop an open-string description of these D-branes. The dynamics of the domain walls is shown to be governed by a Chern-Simons-Higgs theory which, at the quantum level, captures the classical scattering of domain wall solitons.

Keywords: Supersymmetric gauge theory, D-branes, Solitons Monopoles and Instantons.

Contents

1. Introduction

Ten years ago, Polchinski introduced the concept of a D-brane [1]. It is hard to overstate the role that these objects have played in theoretical physics in the intervening time. They have lead to increased understanding of dualities, given impetus to creative new ideas in phenomenology and cosmology and, perhaps most importantly, unpin our understanding of strongly coupled gauge theory through the AdS/CFT correspondence.

The purpose of this paper is to explore the dynamics of D-branes in the rather more mundane setting of field theory, decoupled from gravity and the associated complications. In particular, we will determine the "open-string" description of the dynamics of D-branes in four dimensional $\mathcal{N} = 2$ super QCD. Part of the motivation for this work is to study the string-gauge theory correspondence for gauge theories in the Higgs phase where the strings in question are semi-classical magnetic flux tubes whose worldvolume dynamics is under good control.

A D-brane is defined as a hypersurface on which a string may end. In a field theory, both the brane and the string itself must arise as solitonic type objects. With this definition there are several systems appearing in Nature which can be said to admit D-branes, including superfluid ^{3}He and anti-ferromagnets. However, we may ask for two further requirements from our D-branes so that they are closer in spirit to those appearing in string theory. The first is that the brane houses a U(1) gauge field living on its worldvolume under which the end of the string is charged. The second requirement is that as two D-branes approach, the stretched string gives rise to a new light excitation on its worldvolume which governs the dynamics of the brane. There are at least three examples of non-gravitational field theories admitting D-branes in this stronger sense. In each case, the D-branes are D2-branes:

- $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions. The theory lies in the confining phase. There exist BPS domain walls which are D-branes for the QCD flux tube [2].

- $\mathcal{N} = 2$ super QCD in four dimensions. The theory lies in the Higgs phase. Again there exist BPS domain walls which, this time, are D-branes for the solitonic magnetic vortex [3, 4].

- $\mathcal{N} = (1, 1)$ super Yang-Mills in six dimensions, which can be thought of as the low-energy limit of type iib little string theory. The theory lies in the Coulomb phase. The spectrum of solitons includes an instanton string and a monopole 2-brane. The latter is a D-brane for the former [5].

In string theory there are two methods to determine the dynamics of D-branes. The "closed string" description — which in practice means the supergravity approximation — is analogous to the method used to study soliton scattering in field theory; one simply follows the evolution of the classical bulk configurations under the equations of motion. For our three examples above, this procedure can be carried out explicitly, using the moduli space approximation, only for the second and third cases where the field theory is weakly coupled and the solitons semi-classical.

The second, dual, method that exists in string theory is the "open string" description. Here one neglects the back-reaction of the D-branes on the spacetime fields, concentrating instead on the light states on the D-brane worldvolume. As two D-branes approach, new light states appear arising from string stretched between them. For solitons in field theories, such an open-string description does not usually exist. However, when the solitons are D-branes for some solitonic string, one may wonder whether it is possible to formulate an open string description of the dynamics. For monopoles in six-dimensions, such a description was given in [5] and for domain walls in $\mathcal{N} = 1$ super Yang-Mills an open string description was suggested in [6]. We review both of these theories in section 5.

The goal of this paper is to derive an open string description for the dynamics of domain walls in $\mathcal{N} = 2$ SQCD.[1] We will show that the dynamics is captured by a Chern-Simons-Higgs theory living on the worldvolume of the walls.

The plan of this paper is as follows. In the next section we will quickly review some classical aspects of the four dimensional abelian theory and show how the domain wall is a D-brane for the vortex string. In section 3 we turn to domain wall interactions. After reviewing the "closed-string" moduli space approximation to domain wall dynamics, we show how one can re-derive the results using open-strings. Section 4 deals with domain walls in the non-abelian theory. We end, in section 5, with a discussion.

[1] A review of these domain walls, and their relationships with other solitons, can be found in [7].

2. Domain walls as D-branes

In this section and the next we start by studying the the simplest D-branes in abelian models. The theory is $\mathcal{N}=2$ SQED, consisting of of U(1) vector multiplet and N_f charged hypermultiplets. To describe the soliton solutions we will require only a subset of the fields: the gauge field, a real neutral scalar ϕ and N_f charged scalars q_i. The scalar potential is dictated by supersymmetry,

$$V=\sum_{i=1}^{N_f}(\phi-m_i)^2|q_i|^2+\frac{e^2}{2}\left(\sum_{i=1}^{N_f}|q_i|^2-v^2\right)^2 \tag{2.1}$$

The model depends on several parameters. The gauge coupling e^2 sits in front of the D-term in the potential. In addition, there are real masses m_i for each flavor and a Fayet-Iliopoulos (FI) parameter v^2 which induces a vacuum expectation value (vev) for q_i. When the theory has vanishing masses it enjoys an SU(N_f) flavor symmetry with the q_i transforming in the fundamental representation. Distinct masses explicitly break this to the maximal torus,

$$\mathrm{SU}(N_f)\ \xrightarrow{m_i}\ \mathrm{U}(1)_F^{N_f-1} \tag{2.2}$$

which acts by rotating the phases of the q_i individually, modulo the U(1) gauge group. The theory has N_f isolated vacua $V=0$ given by

$$\phi=m_i\,,\qquad |q_j|^2=v^2\delta_{ij}\qquad\quad \text{for } i=1,\ldots,N_f \tag{2.3}$$

In each vacuum the U(1) gauge symmetry is spontaneously broken and the theory exhibits a gap. The photon has mass $M_\gamma=ev$, while the remaining N_f-1 scalars that are not eaten by the Higgs mechanism have masses $M_q=|m_j-m_i|$ with $j\neq i$.

Classically we may take the strong coupling limit $e^2\to\infty$ of the theory. This decouples the photon, leaving behind N_f-1 complex degrees of freedom. For vanishing masses, these degrees of freedom give rise to the sigma model with target space[2] $T^\star\mathbf{CP}^{N_F-1}$. This arises from the classical Lagrangian through the standard "gauged linear sigma model" technique [9], in which the hypermultiplet fields are constrained by the D-term, subject to gauge identification. Non-zero masses m_i induce a potential on this target space with N_f zeroes. This potential can be shown to be proportional to the length of an appropriate Killing vector on the target space, associated to the flavor symmetry.

At the quantum level, the four-dimensional sigma model is non-renormalizable. Although we only require classical properties of this theory, it is an implicit assumption that a suitable UV completion exists. We shall comment further on this in section 5.

The existence of isolated, gapped vacua ensures the existence of domain walls in the theory. Consider a domain wall which interpolates between vacuum $\phi=m_i$ as $x^3\to-\infty$

[2] If we keep only half the hypermultiplet scalars q_i as in (2.1) and neglect the complex scalars with opposite charge (usually denoted $\tilde{q}_i$) then the target space is the zero section $\mathbf{CP}^{N_f-1}$. Classically it is consistent to restrict attention to the massive $\mathbf{CP}^{N_f-1}$ sigma-model since only these modes are excited in the D-brane solitons. However, in the quantum theory, fermions induce an anomaly in the restricted $\mathbf{CP}^{N_f-1}$ theory [8], and we must consider the full $T^\star\mathbf{CP}^{N_f-1}$ target space.

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and $\phi = m_j$ as $x^3 \to +\infty$. We choose to order the masses as $m_i < m_{i+1}$. Then, for $i < j$, the first order Bogomolnyi type equations describing the domain wall are

$$\partial_3 \phi = -\frac{e^2}{2}\left(\sum_{i=1}^{N_f} |q_i|^2 - v^2\right), \qquad \mathcal{D}_3 q_i = -(\phi - m_i) q_i \tag{2.4}$$

subject to the appropriate boundary conditions as $x^3 \to \pm\infty$. Solutions to these equations describe 1/2-BPS domain walls with tension

$$T_{\text{wall}} = v^2 (m_j - m_i) \tag{2.5}$$

The theory also admits vortex strings. In the i^{th} vacuum, these are supported by the phase of scalar q_i winding in the plane transverse to the string. The first order equations describing a string lying in the x^3 direction are

$$B_3 = e^2(|q_i|^2 - v^2), \qquad \mathcal{D}_1 q_i = i \mathcal{D}_2 q_i \tag{2.6}$$

Solutions to these equations with unit winding number describe 1/2-BPS vortex strings. They have width $L_{\text{vortex}} \sim 1/ev$ and tension,

$$T_{\text{vortex}} = 2\pi v^2 \tag{2.7}$$

Note that in the limit $e^2 \to \infty$, these classical strings become infinitesimally thin.

2.1 The single wall as a D-brane

Let's review some properties of the domain wall in the simplest $N_f = 2$ theory with two vacua. We will show that this wall houses a U(1) gauge field and plays the role of a D-brane for the vortex string [3, 4]

The single wall has two collective coordinates, both Nambu-Goldsone modes [10]. The first is simply the center of mass of the domain wall, X, arising from broken translational invariance. The second comes from the $\mathrm{U}(1)_F$ flavor symmetry (2.2),

$$\mathrm{U}(1)_F : \quad q_1 \to e^{i\theta} q_1 \quad \text{and} \quad q_2 \to e^{-i\theta} q_2 \tag{2.8}$$

In each vacuum either q_1 or q_2 vanishes and $\mathrm{U}(1)_F$ coincides with the gauge action. However in the core of the domain wall, both q_1 and q_2 are excited and $\mathrm{U}(1)_F$ acts non-trivially. Thus the moduli space for a single domain wall is

$$\mathcal{M}_1 \cong \mathbf{R} \times \mathbf{S}^1 \tag{2.9}$$

where $\mathbf{R}$ labels the center of mass while $\mathbf{S}^1$ labels the phase.

The low-energy dynamics of the wall is determined by allowing each of the collective coordinates to vary over the worldvolume. Up to two derivative terms, the dynamics is described by

$$\mathcal{L}_{\text{wall}} = \frac{1}{2} T_{\text{wall}} \int d^3x \ \partial_\mu X \partial^\mu X + \frac{1}{\Delta m^2} \, \partial_\mu \theta \, \partial^\mu \theta \ + \ \text{fermions} \tag{2.10}$$

where $\mu = 0, 1, 2$ labels the domain wall worldvolume and $\Delta m = (m_2 - m_1)$. The periodicity is $\theta \in [0, 2\pi)$. The fermion zero modes are dictated by the fact that the domain wall is 1/2-BPS and the worldvolume theory therefore preserves four supercharges, or $\mathcal{N} = 2$ supersymmetry in three dimensions. They are neutral and do not couple to X or θ at the two-derivative level.

The periodic scalar θ on the worldvolume may be exchanged in favor of a U(1) gauge field $F_{\mu\nu}$ by the usual duality construction,

$$4\pi T_{\rm wall}\, \partial_\mu \theta = \epsilon_{\mu\nu\rho} F^{\nu\rho} \tag{2.11}$$

The normalization is fixed by the requirement that $\int {}^\star dF \in 4\pi\mathbf{Z}$ as dictated by Dirac. This allows us to rewrite the theory on the domain wall as a free $\mathcal{N} = 2$ U(1) gauge theory in $d = 2 + 1$ dimensions,

$$\mathcal{L}_{\rm wall} = \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2g^2}\, \partial_\mu\psi\, \partial^\mu\psi \ + \ \text{fermions} \tag{2.12}$$

where the 3d gauge coupling is given by

$$\frac{1}{g^2} = \frac{T_{\rm wall}}{4\pi^2 v^4} \tag{2.13}$$

The scalar field ψ in (2.12) has been canonically normalized in a manner that befits the superpartner of the gauge field. It is related to the center of mass X by the scaling $\psi = 2\pi v^2 X = T_{\rm vortex} X$. This normalization is the first hint that the domain wall knows about the vortex.

The simplest way to see that the domain wall is a D-brane for the vortex string is to return to the original X and θ collective coordinates in which we may find a "BIon" spike[3] solution on the worldvolume, describing a string emanating from the domain wall [11, 12]. This BIon solution preserves 1/2 of the supersymmetries on the wall and is given by,

$$\Delta m\, X + i\theta = \log \Delta m(x^1 + ix^2) \tag{2.14}$$

Since $\theta \to \theta + 2\pi$ as we wind once around the origin of the string at $x^1 = x^2 = 0$, the dualization (2.11) ensures that the end of the string gives rise to a radial electric field on the worldvolume: $F_{0r} \neq 0$. This means the end of the string is electrically charged under the U(1) gauge field on the wall.

The above discussion is from the perspective of the domain wall worldvolume. We can also describe the vortex string ending on the domain wall from the bulk perspective. The Bogomolnyi equations describing this 1/4-BPS configuration are a combination of the domain wall equations (2.4) and the vortex equations (2.6) and were first derived in [4]

$$\partial_1\phi = B_1 \ \ , \ \ \partial_2\phi = B_2 \ \ , \ \ \partial_3\phi = B_3 - e^2\left(\sum_{i=1}^{N_f} |q_i|^2 - v^2\right)$$
$$\mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \ \ , \ \ \mathcal{D}_3 q_i = (\phi - m_i) q_i \tag{2.15}$$

[3]The full dynamics of the domain wall is governed by the Born-Infeld action with the truncation to two derivatives given by (2.12). The BIon spike (2.14) has the property that it remains a solution of the full non-linear Born-Infeld theory [3].

Analytic solutions describing a string ending on the domain wall are known only in the $e^2 \to \infty$ limit [3, 13]. At finite e^2 there is a binding energy between the string and the domain wall [14] (see also [15]). This finite, negative contribution to the total energy scales as $\mathcal{E}_{\text{binding}} \sim -\Delta m/e^2$. In what follows we shall discuss the four-dimensional theory in the $e^2 \to \infty$ limit in which this binding energy evaporates.

3. D-brane interactions

In this section we study the interaction of two or more domain walls. Let's begin by reviewing what's known about the classical scattering of domain walls, akin to bulk or "closed string" calculations in string theory.

To study the scattering of multiple domain walls, we need to look at a theory with $N_f \geq 3$ vacua. We start with $N_f = 3$ and study the system of domain walls interpolating between the vacuum $\phi = m_1$ at $x^3 \to -\infty$ and the vacuum $\phi = m_3$ as $x^3 \to +\infty$. This system of domain walls was studied in [16, 17]. The general solution can be thought of as two domain walls, separated by a modulus R. The left hand wall has tension $T_1 = (m_2 - m_1)v^2$, while the right hand wall has tension $T_2 = (m_3 - m_2)v^2$. In between the two walls, the fields approximate the vacuum $\phi = m_2$. In addition to their position collective coordinate, each wall also carries an independent phase collective coordinate, both Goldstone modes arising from the $\mathrm{U}(1)_F^2$ flavor symmetry of the theory.

In the moduli space approximation, the low-energy dynamics of domain walls is described by the $d = 2+1$ dimensional, $\mathcal{N} = 2$ sigma-model with target space given by the domain wall moduli space which is known to be [16]

$$\mathcal{M}_2 = \mathbf{R} \times \frac{\mathbf{R} \times \tilde{\mathcal{M}}_{\text{cigar}}}{\mathcal{G}}$$

Here the first two $\mathbf{R}$ factors parameterize the center-of-mass and overall phase of the soliton respectively. All interesting dynamics is encoded in $\tilde{\mathcal{M}}_{\text{cigar}}$, the two-dimensional relative kink moduli space. The quotient by the discrete group $\mathcal{G}$ acts only on the $\mathbf{S}^1$ fiber of the cigar. For generic domain wall tensions, $\mathcal{G} = \mathbf{Z}$. However, when $T_1 = T_2$, the second $\mathbf{R}$ factor collapses to $\mathbf{S}^1$, and $\mathcal{G} \cong \mathbf{Z}_2$.

The metric on $\mathcal{M}_{\text{cigar}}$ was computed analytically in [17] in the $e^2 \to \infty$ limit. (It does not differ substantially from the metric at finite e^2 which was computed numerically in [18]). The metric is smooth at the origin and asymptotically, as $R \to \infty$, differs from the flat metric on the cylinder by exponentially suppressed corrections. However, the metric

Figure 1: The relative moduli space of two domain walls is a cigar.

is not needed to determine the crude features of domain walls scattering. Motion in the moduli space which rounds the tip of the cigar (denoted in the figure) corresponds to two approaching domain walls which collide in finite time and rebound with their phases exchanged. In particular, the cigar moduli space captures the most important feature of the domain wall dynamics: the walls cannot pass.

Note that although the final result for the domain wall dynamics is expressed in terms of a worldvolume theory, this is very much a "bulk" calculation, with the moduli space $\mathcal{M}_2$ simply encoding the interactions of the bulk fields in a concise geometrical form.

3.1 The open string description

We will now describe a different realization of the domain wall dynamics that doesn't involve solutions to the bulk field equations, but instead arises by integrating in open strings stretched between the two domain walls.

The open string description starts by considering two, free, domain walls with tensions T_1 and T_2. Their low-energy dynamics is governed by two copies of (2.12)

$$\mathcal{L} = \sum_{a=1}^{2} \frac{1}{4g_a^2} F_{\mu\nu}^{(a)} F^{(a)\,\mu\nu} + \frac{1}{2g_a^2}\, \partial_\mu \psi^{(a)} \partial^\mu \psi^{(a)} + \text{fermions} \tag{3.1}$$

where $1/g_a^2 = T_a/4\pi^2 v^4$. Interactions between the domain walls arise from the stretched, light vortex strings. Usually, one shouldn't attempt to integrate in solitonic states in a field theory; we shall discuss the regime of validity of this procedure shortly. For now, let us ask what state the zero mode of the string gives rise to in the worldvolume theory. In principle one could quantize the theory on the vortex string with suitable Dirichlet boundary conditions. (For example, a gauged linear sigma model description of the vortex string theory was presented in [19] to which the techniques of [20] can be applied). Here we determine the open string state using simple consistency arguments.

From the discussion of the previous section, we know that the open vortex string has charge $(-1, +1)$ under the gauge fields $(A_\mu^{(1)}, A_\mu^{(2)})$ and gives rise to a state with mass $T_{\text{vortex}}(X^{(2)} - X^{(1)}) = \psi^{(2)} - \psi^{(1)}$. Since this state is BPS, preserving 1/2 of the four supercharges on the wall, it should come in short representation of supersymmetry. This may be either a vector multiplet or a chiral multiplet. The former is familiar from type II string theory when identical D-branes approach, giving rise to non-abelian symmetry enhancement. But, in our case, the D-branes are not identical: they have different tension and carry different topological charge. In such cases it is usual for the open string to give rise to a chiral multiplet and I claim this indeed occurs for our domain walls. Note that the virtual interactions of these states must have a rather drastic effect, for they must stop the two domain walls from passing each other. We will now see how this comes about.

The chiral multiplet consists of complex scalar q and a Dirac fermion λ, each with charge one under $A_\mu = A_\mu^{(2)} - A_\mu^{(1)}$. Supersymmetry ensures that they also couple to the separation $\psi = \psi^{(2)} - \psi^{(1)}$ and, in particular, there is a Yukawa coupling on the wall

$$\mathcal{L}_{\text{Yuk}} = \bar{\lambda}\psi\lambda \tag{3.2}$$

This ensures that when the walls are separated with $\psi > 0$, the fermion λ gets a mass. However, as is well known, integrating out a charged, massive, Dirac fermion induces a Chern-Simons term for the gauge field [21, 22]. Similarly, integrating in such a fermion also induces a Chern-Simons term. The low-energy energy effective action for the relative motion of the domain walls is therefore given by the Chern-Simons-Higgs theory, with action

$$\begin{aligned}\mathcal{L} = &\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2g^2}\,\partial_\mu\psi\partial^\mu\psi + \mathcal{D}_\mu q^\dagger \mathcal{D}^\mu q + \kappa\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho}\\ &-\psi^2|q|^2 - \frac{g^2}{2}(|q|^2 - \kappa\psi)^2 + \text{fermions}\end{aligned} \tag{3.3}$$

where the fermionic terms are dictated by supersymmetry and include, among others, the Yukawa coupling (3.2). The "reduced" coupling is given by $1/g^2 = 1/(g_1^2 + g_2^2)$. Most importantly, the Chern-Simons coupling is $\kappa = -1/2$. It arises from integrating in the fermion λ with mass $\psi > 0$. Notice the appearance of the $\kappa\psi$ coupling in the D-term: this is the supersymmetric completion of the Chern-Simons coupling.

The lagrangian (3.3) is the open string description of the relative domain wall dynamics. We wish to study the massless degrees of freedom. However, at first glance it appears that the separation ψ of the D-branes is gapped, with the Chern-Simons coupling giving both the photon and ψ a mass $M \sim \kappa g^2$, resulting in a unique classical vacuum of the theory at $\psi = q = 0$. Quantum effects change this conclusion. Suppose we separate the D-branes by turning on $\psi \neq 0$. There is a classical tadpole since the potential energy is $V = g^2\kappa^2\psi^2/2$. The Chern-Simons coupling κ is renormalized at one-loop by integrating out the Dirac fermion λ. This is given by [21, 22]

$$\kappa_{\text{eff}} = -\frac{1}{2} + \text{sign}(\text{Mass}[\lambda]) = -\frac{1}{2} + \text{sign}(\psi) = \begin{cases} 0 & \psi > 0 \\ -1 & \psi < 0 \end{cases} \tag{3.4}$$

When $\psi > 0$, integrating out the fermion is simply the reverse of the integrating-in procedure we performed to derive the effective action (3.3). Here the effective Chern-Simons coupling vanishes and both the photon A_μ and ψ are massless, reflecting the fact that the D-branes are free to move. However, things are very different when $\psi < 0$. Here the $\kappa_{\text{eff}} \neq 0$ and there are no massless degrees of freedom: the D-branes cannot move into this regime. We see that the open string mode indeed prevents the domain walls from passing.

This method of using Chern-Simons interactions to lift portions of the vacuum moduli space was previously studied in [23, 24], following earlier work on the dynamics of 3d $\mathcal{N} = 2$ theories [25, 26]. Integrating out the chiral multiplet also generates interactions between the massless fields ψ and A_μ on the wall worldvolume. Dualizing the photon back to the relative phase θ, the two derivative terms in the effective Lagrangian are given by

$$\mathcal{L} = \frac{1}{2}\,H(\psi)\;\partial_\mu\psi\,\partial^\mu\psi + \frac{1}{2}\,H(\psi)^{-1}\;\partial_\mu\theta\,\partial^\mu\theta \tag{3.5}$$

where $H(\psi) = 1/g^2$ classically which, at one-loop, is corrected to

$$H(\psi) = \frac{1}{g^2} + \frac{1}{2|\psi|} \tag{3.6}$$

This is the metric on a cigar. As $\psi \to \infty$, the metric is that of a flat cylinder, up to power-law corrections. This is in contrast to the bulk calculation, where the metric differs by exponential corrections. This is not surprising: as we review shortly, the calculations are valid in different regimes and the Kähler potential receives no protection in theories with four supercharges. Nevertheless, the basic features of the domain wall dynamics are correctly captured by the open string calculation. Note that the metric (3.6) is smooth at the origin $\psi = 0$, but cannot be trusted there since higher loop corrections become important. Nevertheless, the results of [26, 24] guarantee that the true metric remains smooth at the origin.

3.2 Multiple domain walls

We may repeat the above procedure for the $N_f - 1$ walls in the U(1) theory with N_f matter fields, interpolating between the two outermost vacua $\phi = m_1$ and $\phi = m_{N_f}$. Each of the $N_f - 1$ walls has a different tension, $T_k = (m_{k+1} - m_k)v^2$. The moduli space of domain wall solutions was studied in [16, 17]. Once again, solutions exist with arbitrary separations between the walls. However, the ordering of the walls is completely fixed, with $X^{(k+1)} \geq X^{(k)}$ where $X^{(k)}$ is the position of the k^{th} domain wall. The moduli space metric remains smooth as domain walls coincide.

The open string description arises by integrating in chiral multiplets between pairs of neighboring domain walls, each generating a Chern-Simons term. The theory on the domain wall worldvolume is therefore given by 3d $\mathcal{N} = 2$ Maxwell-Chern-Simons-Higgs theory, with gauge group $\prod_{k=1}^{N_f-1} \mathrm{U}(1)_k$ and Chern-Simons couplings

$$-\frac{1}{2}\sum_{k=1}^{N_f-2} (A^{(k+1)} - A^{(k)}) \wedge (F^{(k+1)} - F^{(k)}) \tag{3.7}$$

together with $N_f - 2$ chiral multiplets, with charge -1 under $\mathrm{U}(1)_k$ and charge $+1$ under $\mathrm{U}(1)_{k+1}$ for $k = 1, \ldots, N_f - 2$. The fermions in these chiral multiplets have mass $M_k = (\psi^{(k+1)} - \psi^{(k)})$. Upon integrating out the fermions, the Chern-Simons couplings vanish only in the regime $\psi^{(k+1)} > \psi^{(k)}$, mimicking the correct ordering of the domain walls.

3.3 Regimes of validity

Although the two computations described above lead to the same crude physics of domain wall scattering, namely their ordering, the details disagree. Most notably, the leading order velocity interactions are exponentially suppressed in domain wall separation in the bulk computation, while they are only suppressed by power-law in the open-string computation. We shall now examine the regime of validity of these two different approaches.

The bulk calculation was classical, with the scattering of domain walls determined by the field equations. The sigma-model in four dimensions is not renormalizable and new UV degrees of freedom must be introduced, giving expected corrections to the kinetic terms of the form,

$$|\mathcal{D}q|^2 \left(1 + \frac{1}{v^2}|\mathcal{D}q|^2 + \cdots\right) \tag{3.8}$$

We insist that these corrections are negligible. The domain wall interpolating between the $i^{\rm th}$ and $j^{\rm th}$ vacua has width $\Delta m = |m_i - m_j|$, and these higher derivative corrections may be ignored when:

$$\text{Bulk Criterion:} \quad \frac{\Delta m}{v} \ll 1 \tag{3.9}$$

For the open string description to be valid, we require that the stretched vortex string, which has mass $M \sim v^2 R$, is the lightest degree of freedom. Furthermore, excited states of the string should decouple. For the vortex strings, there are two such excitations: oscillations have energy $\mathcal{E}_{\rm osc} \sim v$, while internal excitations have energy $\mathcal{E}_{\rm int} \sim \Delta m$. If we further impose that the separation R between the D-branes is greater than their width $L_{\rm wall} = 1/\Delta m$, then we learn that the open string description is valid in the opposite regime,

$$\text{Open String Criterion:} \quad \frac{v}{\Delta m} \ll 1 \tag{3.10}$$

so that the string oscillator states are lighter than the internal modes. Note that it follows that the open string description is only valid when $R \ll 1/v = 1/T^{1/2}_{\rm vortex}$, which we may call "sub-stringy" distance scales.

String coupling. Since the domain walls are D-branes for the vortex string, it is natural to ask if they have the typical $1/g_s$ tension formula familiar from type II string theory. Defining the inverse string tension $\alpha' = 2\pi/T_{\rm vortex}$, we can write

$$T_{\rm wall} = v^2 \Delta m = \frac{\Delta m}{v} \frac{1}{\alpha'^{3/2}} \tag{3.11}$$

which suggests that the string coupling constant g_s may be

$$g_s = \frac{v}{\Delta m} \tag{3.12}$$

I do not know if the ratio $v/\Delta m$ can indeed be interpreted as the coupling of the vortex string. For small e^2, the vortices are semi-classical objects which are known to reconnect with probability one [27]: at weak gauge coupling, the vortex strings are not weakly coupled. Nonetheless, as the coupling is increased, this semi-classical treatment is no longer valid and the vortices may potentially tunnel past each other. In the sigma-model limit considered in this paper (defined by taking $e^2 \to \infty$ in the classical Lagrangian) the vortices are infinitesimally thin and cannot be treated as semi-classical objects. In this regime, a string coupling of the form (3.12) is not implausible. It's interesting to note that with this interpretation, the open string description is valid at $g_s \ll 1$ as expected.

3.4 A further generalization: complex masses

So far we have described the moduli space of domain walls in the theory with purely real masses m_i. In general these masses may be complex and still be consistent with $\mathcal{N} = 2$ supersymmetry in four dimensions. Turning on a small, imaginary part for each

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mass induces an attractive force between the domain walls. This is most simply seen by studying the BPS tension formula, which now reads

$$T_{\rm wall} = v^2|m_i - m_j| \tag{3.13}$$

With $m_i - m_j$ complex, Pythagoras implies an attractive force between two walls provided that a BPS bound state exists. From the perspective of the moduli space approximation, this attractive force can be described by a potential on the domain wall moduli space which is proportional to the modulus of a Killing vector arising from the $\mathrm{U}(1)_F^{N_f-1}$ flavor symmetries. Complex masses also allow for new 1/4-BPS dyonic domain walls [28] and domain wall junctions [29].

In the open string description, this attractive force arising from complex masses is captured by a suitable FI parameter. This is most simply described for the case of two walls. We may use the $\mathrm{U}(1)_R$ symmetry of the four dimensional theory to set $\mathrm{Im}(m_3 - m_1) = 0$. Then the FI parameters are given by

$$\zeta = \frac{|\,\mathrm{Im}(m_2 - m_1)|}{2\pi} = \frac{|\,\mathrm{Im}(m_3 - m_2)|}{2\pi} \tag{3.14}$$

These appear in the scalar potential of the open string theory (3.3) describing the domain wall dynamics, which now reads

$$V_{\rm wall} = \psi^2|q|^2 + \frac{g_1^2 + g_2^2}{2}(|q|^2 - \kappa\psi - \zeta)^2 \tag{3.15}$$

As the two walls are separated $\psi \to \infty$, the potential is minimized by $|q|^2 = 0$ and flattens out to a constant (recall that, after the one-loop correction, $\kappa = 0$),

$$V_{\rm wall} \to \frac{(g_1^2 + g_2^2)\,\zeta^2}{2} \tag{3.16}$$

which, to leading order in ζ, is equal to the binding energy between the domain walls arising from (3.13). The unique, quantum ground state of the theory is given by $\psi = 0$ and $|q|^2 = \zeta$. This corresponds to the supersymmetric bound state of two domain walls. The open string mode becomes tachyonic at $\psi \sim g^2\zeta$ and condenses in the vacuum.

4. D-branes in non-abelian theories

So far we have discussed domain walls in abelian gauge theories which have fixed ordering in space. We now turn to the more general non-abelian theories where the ordering of domain walls is more complicated. Nevertheless, we shall see that it is once again captured by the quantum dynamics of a Chern-Simons theory on the wall.

We work with a four dimensional $\mathcal{N} = 2$, $\mathrm{U}(N_c)$ gauge theory with N_f matter multiplets in the fundamental representation. As in section 2, we need only work with a subset of the fields: a real adjoint scalar ϕ and N_f complex fundamental scalars q_i, $i = 1, \ldots, N_f$. The scalar potential is dictated by supersymmetry,

$$V = \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i + \frac{e^2}{2}\,\mathrm{Tr}\left(\sum_{i=1}^{N_f} q_i \otimes q_i^\dagger - v^2\, 1_N\right)^2 \tag{4.1}$$

This potential has a large number of distinct, isolated, vacua. There are supersymmetric vacua with $V = 0$ only if $N_f \geq N_c$ and we assume this is the case. Each vacuum is determined by a choice of N_c distinct elements from a set of N_f

$$\Xi = \{\xi(a) : \; \xi(a) \neq \xi(b) \text{ for } a \neq b\} \tag{4.2}$$

where $a = 1, \ldots, N_c$ runs over the color index, and $\xi(a) \in \{1, \ldots, N_f\}$. Choose $\xi(a) < \xi(a+1)$. Both terms in the potential vanish by setting

$$\phi = \mathrm{diag}(m_{\xi(1)}, \ldots, m_{\xi(N_c)})\,, \qquad q^a{}_i = v\delta^a{}_{i=\xi(a)} \tag{4.3}$$

The number of vacua of this type is

$$N_{\rm vac} = \binom{N_f}{N_c} = \frac{N_f!}{N_c!(N_f - N_c)!} \tag{4.4}$$

Each vacuum has a mass gap in which there are N_c^2 gauge bosons with $M_\gamma^2 = e^2v^2 + |m_{\xi(a)} - m_{\xi(b)}|^2$, and $N_c(N_f - N_c)$ quark fields with mass $M_q^2 = |m_{\xi(a)} - m_i|^2$ with $i \notin \Xi$.

Domain walls interpolate between a given vacuum Ξ_- at $x^3 \to -\infty$ and a distinct vacuum Ξ_+ at $x^3 \to +\infty$. The first order non-abelian domain wall equations are the obvious generalization of (2.4),

$$\mathcal{D}_3\phi = -\frac{e^2}{2}\left(\sum_{i=1}^{N_f} q_i \otimes q_i^\dagger - v^2\, 1_N\right), \qquad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{4.5}$$

Solutions to these equations have tension given by,

$$T_{\rm wall} = v^2[\mathrm{Tr}\,\phi]_{-\infty}^{+\infty} = v^2 \sum_{i\in\Xi_+} m_i - v^2 \sum_{i\in\Xi_-} m_i \tag{4.6}$$

A strict classification of domain wall systems in this model requires a specification of vacua Ξ_- and Ξ_+ at left and right infinity. However, this leads to a bewildering array of possibilities, since the number of vacua (4.4) grows exponentially with N_f. A coarser classification of domain wall systems was introduced in [14] which contains the important information about the topological sector, without specifying the vacua completely. The first step is to introduce the N_f-vector,

$$\vec{m} = (m_1, \ldots, m_{N_f}) \tag{4.7}$$

We can then write the tension of the domain wall as

$$T_{\rm wall} = v^2\, \vec{g}\cdot\vec{m} \tag{4.8}$$

which defines a vector $\vec{g}$ that contains entries 0 and ± 1 only. In analogy to the classification of monopoles [30], we further decompose this vector as

$$\vec{g} = \sum_{i=1}^{N_f} n_i\, \vec{\alpha}_i \tag{4.9}$$

where $n_i \in \mathbf{Z}$ and $\vec{\alpha}_i$ are the simple roots of $su(N_f)$,

$$\begin{aligned} \vec{\alpha}_1 &= (1,-1,0,\dots,0) \\ \vec{\alpha}_2 &= (0,1,-1,\dots,0) \\ \vec{\alpha}_{N_f-1} &= (0,\dots,0,1,-1) \end{aligned} \tag{4.10}$$

The condition that $\vec{g}$ contains only 0's, 1's and -1's translates into the requirement that neighboring n_i's differ by at most one: $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$. Note that in this notation, the domain walls in the abelian theory that we discussed in section 3.2 have topological charge $\vec{g} = (1,0,\dots,0,1)$, or $n_i = 1$ for all i.

4.1 The ordering of domain walls

As in section 3, the bulk description of the domain wall dynamics is studied in the moduli space approximation by examining solutions to the static equations (4.5). Given two vacua Ξ_- and Ξ_+ at left and right infinity, the number of collective coordinates of the interpolating domain wall system is determined by the vector $\vec{g}$ [13, 14]

$$\#\ \text{collective coords} = 2\sum_{i=1}^{N_f-1} n_i \tag{4.11}$$

The interpretation of this result is that there are $N_f - 1$ types of "elementary" domain walls associated to the simple roots $\vec{g} = \vec{\alpha}_i$. The tension of the i^{th} elementary domain wall is given by

$$T_i = v^2(m_{i+1} - m_i) \tag{4.12}$$

A domain wall in the sector $\vec{g}$ decomposes into $N = \sum_i n_i$ elementary domain walls, each with its own position and phase collective coordinate.

The moduli space of solutions was studied in some detail in [13, 31]. Most important is the question of ordering, for in the non-abelian theory it is no longer true that walls can never pass through each other. Domain walls which live in different parts of the flavor group, so that $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$, do not interact and can happily move through each other. When these domain walls are two of many in a topological sector $\vec{g}$, an interesting pattern of interlaced walls emerges, determined by which walls bump into each other and which pass through each other. One finds [31] that the n_i elementary $\vec{\alpha}_i$ walls must be interlaced with the $\vec{\alpha}_{i-1}$ and $\vec{\alpha}_{i+1}$ walls. (The concept of interlacing makes sense because $n_1 = n_{i+1}$ or $n_i = n_{i+1} \pm 1$). The pattern of domain walls in space is depicted in figure 2 where x^3 is plotted horizontally and the vertical position of the domain wall denotes its type. Notice that the $\vec{\alpha}_1$ domain wall is sandwiched between the two $\vec{\alpha}_2$ domain walls which, in turn, are trapped between the three $\vec{\alpha}_3$ domain walls. However, the relative positions of the $\vec{\alpha}_1$ domain wall and the middle $\vec{\alpha}_3$ domain wall are not fixed: these objects can pass through each other.

This concludes the discussion of the bulk dynamics. It is rather simple to reconstruct this ordering from the open string perspective. We start with the free $\mathrm{U}(1)^N$ gauge theory

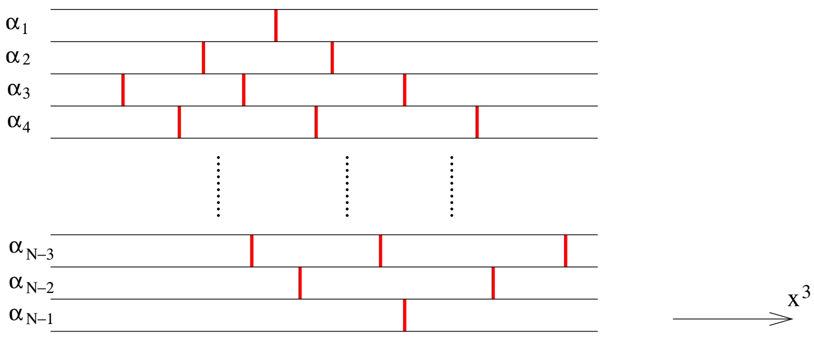

Figure 2: The ordering of many domain walls. The horizontal direction is their position, while the vertical denotes the type of domain wall.

on $N = \sum_i n_i$ domain walls. The ordering described above arises by integrating in chiral multiplets in the manner described in section 3. The restriction that domain wall i lies to the left of domain wall j may be enforced by the introduction of a chiral multiplet with charge $(-1, +1)$ under the two gauge groups. Integrating in this chiral multiplet gives rise once again to a Chern-Simons term of the the form

$$-\frac{1}{2}(A^{(j)} - A^{(i)}) \wedge (F^{(j)} - F^{(i)}) \tag{4.13}$$

and the associated supersymmetric completion in the D-term. The arguments of the previous section generalize trivially. However, there's a small subtlety that does not arise in the case of abelian domain walls, since the i^{th} and j^{th} walls need not necessarily be neighbours. For example, the $\vec{\alpha}_1$ domain wall in the figure must must have two chiral multiplets, one charged under each of the $\vec{\alpha}_2$ gauge fields. It doesn't interact directly with the $\vec{\alpha}_3$ wall. But only one of the $\vec{\alpha}_2$ walls is a neighbour of the $\vec{\alpha}_1$ domain wall; on the other side sits the $\vec{\alpha}_3$ domain wall. One may worry that no BPS string can pass through the $\vec{\alpha}_3$ domain wall, to end on the more distant $\vec{\alpha}_2$ wall. In [14] a detailed study was performed on which strings can end on which walls, and which strings may pass through walls unaffected. One may check that there is always a BPS string connecting a $\vec{\alpha}_i$ domain wall with the two closest $\vec{\alpha}_{i+1}$ and $\vec{\alpha}_{i-1}$ domain walls, consistent with the conjecture of a chiral multiplet arising in from the open string.

There's one further difference with domain walls in the non-abelian theory: we now have domain walls with the same tension (4.12) that may be possibly be classified as "identical". Could strings stretched between the $\vec{\alpha}_i$ domain walls give rise to a non-abelian symmetry enhancement: $\mathrm{U}(1)^{n_i} \to \mathrm{U}(n_i)$? To determine this really requires a semi-classical quantization of the vortex string with Dirichlet boundary conditions, but it appears that non-abelian symmetry enhancement does not occur for these domain walls.[4] The key point is that identical walls are never neighbours. A string connecting the walls only exists (at

[4]An attempt at non-abelian symmetry enhancement on the domain wall worldvolume was made in [32], although not through the use of open strings. The authors consider degenerate masses, resulting in a non-abelian global symmetry on the wall. At the linearised level, this may be dualised for a gauge symmetry, but it is unclear if this gives rise to a non-abelian gauge symmetry at the non-linear level.

finite e^2) if it is threaded with a confined monopole [33, 14]. One may check, using the rules described in [14], that these strings are not BPS for all possible positions of the intervening walls.

To summarize: the open string theory on the walls in topological sector $\vec{g}$ is given by $U(1)^{\sum_i n_i}$ gauge theory, with chiral multiplets linking the closest $\vec{\alpha}_i$ and $\vec{\alpha}_{i\pm1}$ walls, together with the associated Chern-Simons interactions.

5. Discussion

In this paper we have developed the open string description of semi-classical domain walls in a four-dimensional field theory. It is instructive to compare the results against other field theoretic D-branes.[5] As mentioned in the introduction, there are at least two other examples of D-branes in theories without gravity for which an open string description is known. In both cases, the objects are D2-branes. Let's briefly review:

Monopoles in little string theory. $\mathcal{N} = (1,1)$ super Yang-Mills in six dimensions can be thought of as the low-energy limit of type iib little string theory. When the theory lives on the Coulomb branch, the spectrum of solitons includes a monopole 2-brane. This is a D-brane for the instanton string [5, 36].

The open string description of the monopole dynamics has much in common with the domain walls described in this paper. (See, for example, [37] for a recent account of the relationship between monopoles and domain walls). The moduli space of a single monopole is $\mathbf{R}^3 \times \mathbf{S}^1$. Since the monopole has three-dimensional worldvolume, the periodic scalar may be dualised in favor of a U(1) gauge field living on the brane. The instanton string may terminate on the monopole 2-brane, where its end is electrically charged under this gauge field.

BPS monopoles in an SU(N) gauge group are classified by a magnetic charge vector $\vec{g} = \sum_i n_i \vec{\alpha}_i$, where $\vec{\alpha}_i$ are the simple roots of $su(N)$, and $n_i \in \mathbf{Z}^+$ [30]. Integrating in the open instanton strings[6] stretched between monopoles enhances the free $\mathcal{N} = 4$, $U(1)^{\sum_i n_i}$ gauge theory on the worldvolume to $\prod_{i=1}^{N-1} U(n_i)$ with bi-fundamental hypermultiplets transforming in the $(\bar{\mathbf{n}}_i, \mathbf{n}_{i+1})$ representation. The quantum dynamics of these theories were explored in [38 − 40] where it was shown that the Coulomb branch metric of the three dimensional gauge theory coincides with the metric of the appropriate monopole moduli space. In this case, the open string description agrees with the closed string description up to the two-derivative level, despite being valid in different regimes. As usual, this can be traced to a non-renormalization theorem (essentially the lack of deformations of the appropriate hyperKähler metrics).

Domain walls in $\mathcal{N} = 1$ super Yang-Mills. Unlike the two previous examples, domain walls in $\mathcal{N} = 1$ SU(N) super Yang-Mills cannot be treated semi-classically. Nonetheless,

[5]Other analogies between field theoretic solitons and D-branes have been discussed in [34, 35].

[6]To my knowledge, there has been no explicit semi-classical computation of the open string spectrum by endowing instanton strings with Dirichlet boundary conditions.

the fact that they are BPS objects [41] means that they provide one of the few handles on the strongly coupled, infra-red regime of the theory. Using M-theory techniques, Witten showed that the domain wall is a D-brane for the QCD string [2].

The open string description of the dynamics of k domain walls was determined by Acharya and Vafa [6] by performing a geometric transition in IIA string theory. The low-energy dynamics of the walls are described by a 3d U(k) gauge theory with $\mathcal{N} = 1$ supersymmetry. There is an adjoint multiplet, describing the separation of the walls, and a Chern-Simons term at level N. In a recent impressive calculation, Armoni and Hollowood have shown how the attractive force between domain walls is captured by this open string description at two loops [42]. Although the open string description agrees qualitatively with the bulk calculation, it differs in the details. For example, the leading order force arising from integrating out the open string zero mode is power-law in the separation, whereas the bulk calculation gives rise to an exponentially suppressed force. This is rather similar to what we found above, with the open string kinetic term (3.5) deviating by power-law corrections from the flat metric, rather than exponential corrections.

An underlying closed string description? In each of these three examples, the quantum open-string description captures the classical bulk dynamics of the solitons. The degree to which the two quantitatively agree is determined by the amount of supersymmetry and the associated non-renormalization theorems. Of course, such agreement is commonplace within string theory and is often summarized by the familiar annulus diagram in which a closed string propagating at tree level between the two branes may be re-interpreted as a vacuum loop of an open string stretched between the branes.

Both the four-dimensional and six-dimensional super Yang-Mills theories described in this discussion have an underlying non-critical string description. For the former, it is the usual 't Hooft expansion at large N; for the latter it is the type iib little string theory propagating on the decoupled NS5-brane. In each of these examples, the bulk description of D-brane scattering could well be called a "closed-string" description. But what about the $\mathcal{N} = 2$ four-dimensional theory that is the main focus of this paper? Is there also an underlying non-critical string theory? The theory, when considered in the $e^2 \to \infty$ limit, is a non-renormalizable massive sigma model. It is interesting to speculate that there may be an underlying little string theory of the vortex flux tubes providing the UV completion of the theory. The existence of the open string description of the worldvolume dynamics could then be interpreted in the familiar framework of open-closed string duality.

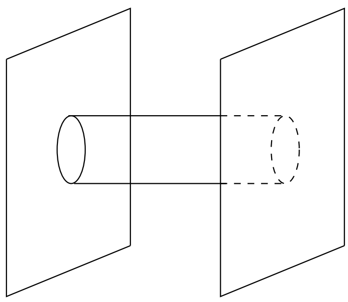

Figure 3.

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